Supporting information for "Covariate-adaptive historical control borrowing Bayesian design" by Huaqing Jin, Mi–Ok Kim, Aaron Scheffler and Fei Jiang

A The estimated posterior variances of μ_0 under different prior specifications.

A.1 General estimators

As shown in (2.1), the posterior distribution of μ_0 satisfies,

$$\mathcal{P}_{\mu_0}\{\mu_0(\mathbf{X})|\phi_0, \tau(\cdot), \mathcal{D}_n\} \propto \prod_{i=1}^{n} \left(\mathcal{L}\{\mu_0(\mathbf{X}_i), \phi_0, Y_i\} \exp\left[-\frac{\{\mu_0(\mathbf{X}_i) - \theta_0(\mathbf{X}_i)\}^2 \tau(\mathbf{X}_i)}{2}\right]\right)^{(1-Z_i)\Delta_i(\mathbf{X})}.$$

When $\mu_0(\cdot)$ is considered as a smooth function, we replace Δ_i by $K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})$ and compute the posterior variance as

$$\begin{aligned} & \widehat{\text{var}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \widehat{\boldsymbol{\tau}}(\mathcal{D}_n), \mathcal{D}_n\} \\ = \;& E\{\mu_0(\mathbf{X})^2|\widehat{\phi}_0, \widehat{\boldsymbol{\tau}}(\mathcal{D}_n), \mathcal{D}_n\} - E\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \widehat{\boldsymbol{\tau}}(\mathcal{D}_n), \mathcal{D}_n\}^2, \end{aligned} \tag{A.1}$$

where the expectation is taking with respect to the density

$$\mathcal{P}'_{\mu_0}\{\mu_0(\mathbf{X})|\phi_0, \tau(\cdot), \mathcal{D}_n\} \propto \prod_{i=1}^{n} \left(\mathcal{L}\{\mu_0(\mathbf{X}), \phi_0, Y_i\} \exp\left[-\frac{\{\mu_0(\mathbf{X}) - \theta_0(\mathbf{X}_i)\}^2 \tau(\mathbf{X}_i)}{2}\right]\right)^{(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i-\mathbf{X})}.$$

On the other hand, when using a flat prior of μ_0 in the reference model, the corresponding posterior density is

$$\mathcal{P}_{\text{ref}}\{\mu_0(\mathbf{X})|\phi_0, \mathcal{D}_n\} \propto \prod_{i=1}^{n} [\mathcal{L}\{\mu_0(\mathbf{X}_i), \phi_0, Y_i\}]^{(1-Z_i)\Delta_i(\mathbf{X})}.$$

We compute $\widehat{\text{var}}_{\text{ref}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \mathcal{D}_n\}$ based on (A.1), while the expectation is taking with respect to the density

$$\mathcal{P}'_{\text{ref}}\{\mu_0(\mathbf{X})|\phi_0, \mathcal{D}_n\} \propto \prod_{i=1}^{n} [\mathcal{L}\{\mu_0(\mathbf{X}), \phi_0, Y_i\}]^{(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i-\mathbf{X})}.$$

A.2 Estimators under the normal likelihood

When $\mathcal{L}$ is a normal likelihood, we have

$$\begin{aligned}
& \mathcal{P}'_{\mu_0}\{\mu_0(\mathbf{X})|\phi_0, \tau(\cdot), \mathcal{D}_n\} \\
\propto & \prod_{i=1}^{n}\left(\mathcal{L}\{\mu_0(\mathbf{X}), \phi_0, Y_i\} \exp\left[-\frac{\{\mu_0(\mathbf{X}) - \theta_0(\mathbf{X}_i)\}^2\tau(\mathbf{X}_i)}{2}\right]\right)^{(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i-\mathbf{X})}, \\
\propto & \prod_{i=1}^{n}\left(\exp\left[-\frac{\{Y_i - \mu_0(\mathbf{X})\}^2}{2\phi_0^2} - \frac{\{\mu_0(\mathbf{X}) - \theta_0(\mathbf{X}_i)\}^2\tau(\mathbf{X}_i)}{2}\right]\right)^{(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i-\mathbf{X})}, \\
= & \exp\left[-\frac{\mu_0^2(\mathbf{X})}{2}\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\left\{\frac{1}{\phi_0^2} + \tau(\mathbf{X}_i)\right\}\right] \\
& \times \exp\left[\mu_0(\mathbf{X})\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\left\{\frac{Y_i}{\phi_0^2} + \tau(\mathbf{X}_i)\theta_0(\mathbf{X}_i)\right\}\right].
\end{aligned}$$

Let

$$\begin{aligned}
A &= \sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\left\{\frac{1}{\phi_0^2} + \tau(\mathbf{X}_i)\right\}, \\
B &= \mu_0(\mathbf{X})\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\left\{\frac{Y_i}{\phi_0^2} + \tau(\mathbf{X}_i)\theta_0(\mathbf{X}_i)\right\},
\end{aligned}$$

we can see that

$$\mathcal{P}'_{\mu_0}\{\mu_0(\mathbf{X})|\phi_0, \tau(\cdot), \mathcal{D}_n\} \propto \exp\left\{-\frac{\mu_0^2(\mathbf{X})}{2}A + \mu_0(\mathbf{X})B\right\} \propto \exp\left[-\frac{A}{2}\left\{\mu_0(\mathbf{X}) - \frac{B}{A}\right\}^2\right]$$

Therefore, it is clearly that the posterior distribution of $\mu_0(\mathbf{X})$ is a normal distribution with mean

$$\frac{B}{A} = \frac{\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\{Y_i\phi_0^{-2} + \tau(\mathbf{X}_i)\theta_0(\mathbf{X}_i)\}}{\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\{\phi_0^{-2} + \tau(\mathbf{X}_i)\}}$$

and variance

$$\frac{1}{A} = \frac{1}{\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\{\phi_0^{-2} + \tau(\mathbf{X}_i)\}}.$$

Replacing $\phi_0, \tau(\cdot)$ by their estimators $\widehat{\phi}_0, \widehat{\tau}(\cdot)$, we have

$$\widehat{\text{var}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \widehat{\boldsymbol{\tau}}(\mathcal{D}_n), \mathcal{D}_n\} = \frac{1}{\sum_{i=1}^{n}(1-Z_i)K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\{\widehat{\phi}_0^{-2} + \widehat{\tau}(\mathbf{X}_i)\}},$$

Similar derivations based on $\mathcal{P}'_{\text{ref}}\{\mu_0(\mathbf{X})|\phi_0, \mathcal{D}_n\}$ yields

$$\widehat{\text{var}}_{\text{ref}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \mathcal{D}_n\} = \frac{\widehat{\phi}_0^2}{\sum_{i=1}^n (1 - Z_i) K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})}.$$

Therefore, when $\mathcal{L}$ is a normal likelihood, we obtain

$$R_n(\mathbf{X}) = \frac{\widehat{\text{var}}_{\text{ref}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \mathcal{D}_n\}}{\widehat{\text{var}}\{\mu_0(\mathbf{X})|\widehat{\phi}_0, \widehat{\boldsymbol{\tau}}(\mathcal{D}_n), \mathcal{D}_n\}} = \frac{\sum_{i=1}^n (1 - Z_i) K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})\{1 + \widehat{\phi}_0^2 \widehat{\tau}(\mathbf{X}_i)\}}{\sum_{i=1}^n (1 - Z_i) K_{\mathbf{H}}(\mathbf{X}_i - \mathbf{X})}.$$